proof of the fact, anticipated by Gauss and announced by Ruffini, that it is impossible to express the solution of a general equation as algebraic functions of the coefficients when the degree exceeds the fourth." As the first publication by Galois on the theory of groups appeared after the death of Abel it is difficult to see how his proof could have been based upon the discoveries of Galois. At any rate the sentence as quoted above would naturally lead the reader to think that Abel based his work upon work done earlier by Galois and hence it is unsatisfactory in its present form.

We shall refer here to only one more sentence which seems to convey an incorrect impression in regard to a historical fact of considerable mathematical interest. This sentence appears in the article on the Italian mathematician *Cardan,* volume 4, page 536, and is as follows: "The publication of the *Ars Magna* stimulated mathematical research and hastened the general solution of the biquadratic equation, of which Cardan himself had solved special cases." If we read this statement in the light of the fact that Ferrari's general solution of the biquadratic equation actually appeared in the *Ars Magna* it may possibly have some meaning, but it is evident that the beginner would be apt to draw entirely incorrect conclusions therefrom.

The few modifications which have been suggested could scarcely be supposed to be of general interest to mathematics teachers if they did not relate to an excellent work of reference which is very extensively used by college and university students. It is evidently highly desirable that such works be as clear and accurate as possible. We can scarcely expect that the publishers will make special efforts to attain these ends unless the public actually demands them. Hence publicity given to shortcomings, especially where such publicity tends to the discovery of many other important improvements, seems desirable. Such publicity may also tend to inspire caution in the use of even the most reliable works of reference, a caution which needs to be cultivated on the part of most young mathematicians.

NEW RULES OF QUADRATURE.

By P. J. DANIELL, Rice Institute.

The rules given in this paper are developed from Euler's summation formula.[1] This formula has been used in the past chiefly as a means of converting a series into an integral. Nevertheless it has several advantages as a source from which to obtain rules of quadrature. Three such rules are stated here, and the author believes that the second and third are new, while even the first has not received the attention which it deserves.

Rule 1.

$$\int_a^b ydx = h[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n] + \frac{h^2}{12}[y_0' - y_n'] + R_1.$$

[1] Bromwich, *Theory of Infinite Series,* Chap. X, p. 238.

Here the interval (a, b) is divided into n equal parts of length h, $y_0, y_1, y_2 \cdots$ are the ordinates at $a, a+h, \cdots$ and y_0', y_n' are the derivatives of y at a and b. If R_1 denotes the remainder, then

$$R_1 = -\frac{1}{4!}\int_a^b y^{\mathrm{IV}}(x)\varphi_1(x)dx,$$

where $y^{\mathrm{IV}}(x)$ denotes the fourth derivative of y with respect to x, and $\varphi_1(x)$ denotes the function

$$\varphi_1(x) = (x - a - rh)^2(a + rh + h - x)^2, \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this it is only necessary to integrate R_1 by parts four times.

$$\int_{a+rh}^{a+rh+h} \varphi_1(x)dx = h^5\int_0^1 t^2(1-t)^2dt = \frac{h^5}{30}.$$

Then

$$|R_1| \leqq \frac{h^4}{6!}(b-a)\max|y^{\mathrm{IV}}(x)|.$$

Comparison with Simpson's Rule. To find the error in Simpson's rule, let n be even; then by a similar process, taking $n/2$ intervals of length $2h$,

$$\int_a^b ydx = 2h[\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_n] + \frac{4h^2}{12}[y_0' - y_n'] + R_2;$$

where

$$R_2 = -\frac{1}{4!}\int_a^b y^{\mathrm{IV}}(x)\varphi_2(x)dx,$$

and

$$\varphi_2(x) = (x - a - 2rh)^2(a + 2rh + 2h - x)^2, \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0' - y_n']$, we obtain Simpson's rule with remainder R', where $R' = \frac{1}{3}(4R_1 - R_2)$, and

$$\frac{1}{3}\int_{a+2rh}^{a+2r+2h}[\varphi_2(x) - 4\varphi_1(x)]dx = \tfrac{2}{3}h^5\int_0^1[t^2(2-t)^2 - 4t^2(1-t)^2]dt = \tfrac{4}{15}h^5.$$

Then

$$|R'| \leqq 4\frac{h^4}{6!}(b-a)\max|y^{\mathrm{IV}}(x)|.$$

Thus the "maximum error," according to rule 1, is only one quarter of that in Simpson's rule.

Closed Curves. For closed curves Simpson's rule is not applicable except by a separate treatment of different portions. But by an application of rule 1 we can obtain an exceedingly simple rule.

The area of a segment between a chord and the curve, considering the chord as a single interval, will be one twelfth of the square of the chord multiplied by

the change in the tangent of the angle between the curve and the chord. If the chord is small compared to the radius of curvature this change will be approximately the angular change in the direction of the curve. If then we have a number of equal successive chords, the total area between them and the curve will be one twelfth of the square of chord times the total angular change in direction of the curve. For a closed curve this will be 2π, hence we obtain the following:

Rule 2. *Step off equal chords along the curve coming back, if possible, to the starting point. Then if O is any point inside the polygon thus formed,* we have

$$\textit{Area} = \tfrac{1}{2}\,\textit{chord} \times \left[\textit{sum of perpendiculars from O on chords} + \frac{\pi}{3} \times \textit{chord}\right].$$

The first part of this expression is the area of the chord polygon while the latter part is

$$\tfrac{1}{12} \times \text{chord}^2 \times 2\pi = \tfrac{1}{2} \times \text{chord} \times \left[\frac{\pi}{3} \times \text{chord}\right].$$

If the chord polygon is not quite closed we can add to the above expression the area of the remaining sector regarded as a triangle. If the closing chord equals a small fraction λ of the equal chords, then λ times the perpendicular from O on the closing chord is to be added to the sum of the perpendiculars in the rule. Otherwise we may step round again, obtaining the value of twice the area and replacing 2π by 4π. Since the rule is only approximate we may replace $\pi/3$ in the bracket by $1\frac{1}{20}$, thus

$$\textit{Area} = \tfrac{1}{2}\,\textit{chord} \times [\textit{sum of perpendiculars from } 0 + 1\tfrac{1}{20} \textit{ of chord}].$$

Rule 3.

$$\int_a^b y dx = \frac{h}{15}(7y_0 + 16y_1 + 14y_2 + 16y_3 + \cdots + 16y_{2n-1} + 7y_{2n}) + \frac{h^2}{15}(y_0' - y_{2n}').$$

Here the interval is divided into an even number $2n$ of parts.

$$\int_a^b y dx = h(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{h^2}{12}(y_0' - y_{2n}') - \frac{h^4}{6!}(y_0''' - y_{2n}''') + S_1,$$

where

$$S_1 = -\frac{1}{6!}\int_a^b y^{\text{VI}}(x)\psi_1(x)dx, \qquad \psi_1(x) = z^2(z + \tfrac{1}{2}),$$

and

$$z = (a + rh + h - x)(x - a - rh), \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this integrate S_1 by parts six times.

Taking n intervals of length $2h$ each,

$$\int_a^b y dx = 2h(\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{4h^2}{12}(y_0' - y_{2n}') - \frac{16h^4}{6!}(y_0''' - y_{2n}''') + S_2,$$

where

$$S_2 = -\frac{1}{6!}\int_a^b y^{\text{VI}}(x)\psi_2(x)dx, \qquad \psi_2(x) = z_2^2(z_2 + \tfrac{1}{2})$$

and

$$z_2 = (a + 2rh + 2h - x)(x - a - 2rh), \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0''' - y_{2n}''']$ between these two, we obtain rule 3 with remainder $S' = \frac{1}{15}(16S_1 - S_2)$, and

$$\frac{1}{15}\int_{a+2rh}^{a+2rh+2h} [\psi_2(x) - 16\psi_1(x)]dx$$

$$= \frac{2}{15}h^7\int_0^1 [t^2(2-t)^2(\tfrac{1}{2} + 2t - t^2) - 16t^2(1-t)^2(\tfrac{1}{2} + t - t^2)]dt$$

$$= \frac{4}{7 \times 25} \times 2h^7.$$

Hence

$$| S' | \leqq \frac{32}{25}\frac{h^6}{8!}(b - a) \max | y^{\text{VI}}(x)| .$$

Examples. (1) Using Rule 2, to find π by means of the regular polygon of 24 sides inscribed in a circle of unit radius, we have

$$\text{chord} = 2 \sin 7°.5, \qquad \text{and} \qquad \text{perpendicular} = \cos 7°.5.$$

Hence,

$$\pi = \sin 7°.5\,[24 \cos 7°.5 + 21/20 \times 2 \sin 7°.5] = 12 \sin 15° + 1.05(1 - \cos 15°).$$

Using

$$\sin 15° = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}), \qquad \text{and} \qquad \cos 15° = \tfrac{1}{4}(\sqrt{6} + \sqrt{2})$$

we find $\pi = 3.14162$ instead of 3.14159.

(2) To approximate the value of $\log_e 2$. We have

$$\int_1^2 \frac{dx}{x} = \log_e 2 = .69315.$$

Using two intervals we get from $\int_1^2 \frac{dx}{x}$ by Simpson's rule, .69444, with the error .00129; by Rule 1, .69271, with the error .00044; and by Rule 3, .69306, with the error .00009.

(3) To approximate the value of π from $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$. We have

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx = \frac{\pi}{12} + \frac{1}{8}\sqrt{3}.$$

Using two intervals we get the value of π from $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$ by Simpson's rule, 3.14093, with the error .00066; by Rule 1, 3.14178, with the error .00019; and by Rule 3, 3.14161, with the error .00002.

SOME METRICAL PROPERTIES OF THE PENTAHEDROID IN A SPACE OF FOUR DIMENSIONS.

By M. H. SZNYTER, University of California.

The purpose of this paper is to establish for the pentahedroid in a four-dimensional space, theorems similar to those found in ordinary solid geometry, dealing with the tetrahedron. The terms line, point and plane are used with the same significance as in three-dimensional geometry. By hyperplane we shall mean that three-dimensional element which consists of any four points not points of one plane, all points collinear with any two of them or with any two obtained by this process.[1] We shall first develop the simpler theorems, then we shall consider the pentahedroid with its tangent hyperspheres.

THEOREM 1.[2] *Let $A_1A_2A_3A_4A_5$ be a pentahedroid cut by a hyperplane α in such a way that the edge A_1A_2 lies on one side of α and the face $A_3A_4A_5$ on the other side. Then the following cases will appear.*

1. *If α is parallel to the line A_1A_2 and to the plane $A_3A_4A_5$, the section will be a prism.*
2. *If α is parallel to the line A_1A_2 but not to the plane $A_3A_4A_5$, the section will be a truncated prism.*
3. *If α is parallel to the plane $A_3A_4A_5$ but not to the line A_1A_2, the section will be a frustrum of a pyramid.*
4. *If α is parallel neither to the plane $A_3A_4A_5$ nor to the line A_1A_2, the section will be a truncated pyramid.*

For the hyperplane α cuts the tetrahedrons $A_2A_3A_4A_5$ and $A_1A_3A_4A_5$ in planes which cut a triangle from each $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$. Since α passes between the edges A_1A_2 and the edges A_3A_4, A_4A_5 and A_3A_5, it must cut the other three tetrahedrons in planes which cut quadrilaterals from them. The section cut out from the pentahedroid will be a three-dimensional figure having for its boundaries two triangular faces and three quadrilateral faces. In cases 1 and 2 the lateral edges of the section are parallel and the section is prismatic. In case 1 the triangular faces are parallel while in case 2 they are not. Hence the former gives a prism while the latter gives a truncated prism. In cases 3 and 4, the lateral edges of the section will meet, if produced, at the point P which is the point of intersection of A_1A_2 with α, and the section is pyramidal. When α is parallel to plane $A_3A_4A_5$, the triangles $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$ are parallel;

[1] MANNING: *Geometry of Four Dimensions*, p. 24.
[2] *Ibid.*, p. 228.